END TERM: TOPOLOGY BMATH2, WINTER 2026

Total time: 3 Hours. **You may answer all** 5 **problems of** 60 **marks to get maximum** 50. Each problem (for instance, Problem 3) carries 12 marks, where each part of a problem (for instance, Problem 2, part (b)) carries 6 marks. You may use results proved in class without proof, but you need to state the result clearly before using it. If a question specifically asks for the proof of a result covered in class, you must provide a detailed proof. If you wish to use a problem from a homework/assignment, supply its solution too. Without a proper explanation, only partial points/no points will be credited.

(1) Let X, Y be normed linear spaces. Let $T : X \to Y$ be a linear map.
  (a) Prove that T is continuous if and only if T is continuous at $0 \in X$.
  (b) Show that T is continuous if and only if there exists $C > 0$ such that $\|T(x)\| \leq C\|x\|$ for all $x \in X$.

(2) (a) Is the space $\mathbb{R}_K$ (real line with K-topology) Hausdorff? Is it Regular?
  (b) Let F be a nonempty closed subset of a compact Hausdorff space X, and assume that F is not open. Prove that the quotient space obtained from X by identifying F to a single point is homeomorphic to the one-point compactification of $X \setminus F$.

(3) (a) Let X be compact, and $\emptyset \neq \mathcal{F} \subset C(X, \mathbb{R})$ be equicontinuous and pointwise bounded. Then show that there exists a compact set $Y \subset \mathbb{R}$ such that $\mathcal{F} \subset C(X, Y)$.
  (b) Let $\mathcal{F}_1 := \{f \in C([0, 1], \mathbb{R}) \mid f(x) = \sum_{n \geq 1} a_n x^n, |a_n| \leq \frac{1}{2^n} \ \forall n \geq 1\}$. Show that $\mathcal{F}_1$ is compact in $C([0, 1], \mathbb{R})$ with sup norm metric.

(4) Let G be a topological group, and H be a closed subgroup of G.
  (a) Show that xH, for any $x \in G$ is a closed set in G. (Any theorem/lemma you use, you must provide a proof.)
  (b) Show that the quotient map $\pi : G \to G/H$ is open. Here, G/H is equipped with the quotient topology.

(5) Let us recall the definition of *locally compact* topological space. X is called locally compact if for every point $x \in X$, there exists a compact set $C \subset X$ and an open set U such that $x \in U \subset C$.

(a) Let G, H and π be as in Problem (4). Show that if G is locally compact, Hausdorff and G/H is compact, then there exists a compact $K \subset G$ such that $\pi(K) = G/H$.
(b) Let $\mathrm{SL}_2(\mathbb{R})$ be the set of 2×2 matrices with *determinant* 1. Show that $\mathrm{SL}_2(\mathbb{R})$ is locally compact in the subspace topology inherited from the set of 2×2 matrices with Euclidean topology.